Transfer Function Analysis of a Flexible Toroidal Structure

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This paper is concerned with the modeling of a flexible toroidal structure for attitude and structural control studies. A two-dimensional elastic system, consisting of a flexible toroid and a pretensioned membrane, is analyzed in terms of the transfer function from a control torque input to the collocated sensor output. The critical tension for out-of-plane buckling is also determined. The models considered in the paper are simple enough to treat analytically, yet complicated enough to demonstrate the dynamical characteristics of future space structures with a toroidal configuration. It is shown that for such structures, the coupling between two elastic systems results in a transfer function expressed as a combination of an infinite product expansion and an infinite partial fraction expansion. Such a representation should be useful in future studies in reduced-order modeling of flexible toroidal structures.

Introduction

LARGE structures are being considered for future missions in space. This paper is concerned primarily with the development of generic models of such space structures with a toroidal configuration. Emphasis is given to the single-input/single-output transfer function models for attitude and structural control studies.

The modeling and control problems of various generic models of flexible space structures have been investigated in Refs. 1–8. In this paper, we present a two-dimensional model that can be analyzed somewhat easily and still be realistic. The model, a flexible toroid with a pretensioned membrane, may be of academic interest without any practical relevance to future space structures that will consist of many lumped and trusslike subsystems with fairly complex interconnections. It is, however, emphasized that the model considered in this paper is a "qualitative" representation of future space structures with a toroidal configuration.

It is shown that for such structures, the coupling between two elastic systems results in a transfer function expressed as a combination of an infinite product expansion and an infinite partial fraction expansion. Such a representation clearly indicates the complicated dynamic behavior of the model. The significance of this result is that the reduced-order modeling of such an axisymmetric structure needs further investigation from a control design viewpoint. Although a transfer function of distributed parameter systems is often represented as either a product expansion (pole-zero form) or a partial fraction expansion (pole-residue form), a combination of product and partial fraction expansions does not appear to have been considered previously in the open literature.

Gevarter[1] studied modeling and control problems of a spinning, flexible toroidal space station, where spinning was necessary to provide an artificial gravity or for spin stabilization. In this paper, we consider a three-axis, stabilized, toroidal structure with a pretensioned membrane-type reflector surface. Since coupling of bending and torsion is a basic property of the flexible toroid, we briefly discuss the attitude motion of a flexible toroid itself. We then present the modeling of a flexible toroid, as well as a rigid toroid, with a pretensioned membrane in terms of the transfer functions.

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Flexible Toroid

Consider the modeling and control of the flexible toroid shown in Fig. 1. Roll attitude control could be obtained by two identical torquers (e.g., reaction wheels) located at points A and B. Pitch control could be obtained by two identical torquers located at points C and D. Obviously, pitch and roll controls are decoupled and are identical, and so we consider only roll attitude control.

The equations of motion of a flexible inextensional toroid for out-of-plane vibration can be written as[9]

Toroid bending:

$$(EI/R^3)[(1/R)y'''' + \theta''] + (GJ/R^3)[\theta'' - (1/R)y''] + \sigma\ddot{y} = 0 \tag{1}$$

Toroid torsion:

$$(GJ/R^2)[\theta'' - (1/R)y''] - (EI/R^2)[(1/R)y'' + \theta] - J_0\ddot{\theta} = 0 \tag{2}$$

where

$( )' \triangleq \partial( )/\partial\psi$, $(\dot{\ }) \triangleq \partial( )/\partial t$
$y(\psi,t)$ = out-of-plane bending displacement
$\theta(\psi,t)$ = torsional displacement
EI = uniform bending stiffness
GJ = uniform torsional stiffness
σ = mass per unit length
J_0 = polar moment of inertia per unit length
R = radius of toroid

The boundary conditions for pure roll motion at $\psi = 0$ and π are

$$y(\psi,t) = 0$$

$$\theta(\psi,t) = 0$$

$$u(t) = (4EI/R)[(1/R)y''(\psi,t) + \theta(\psi,t)] \tag{3}$$

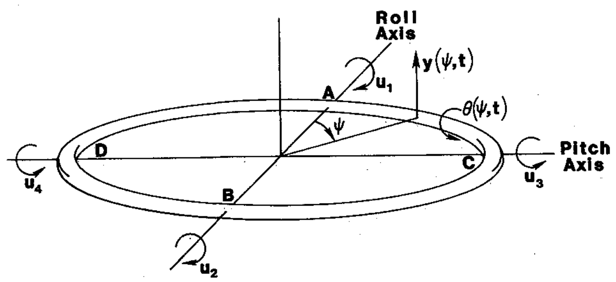

Fig. 1 Flexible toroid with bending-torsion coupling.

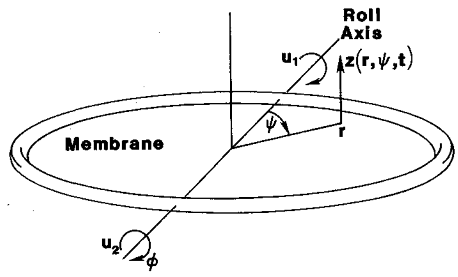

Fig. 2 Rigid toroid with pretensioned membrane.

where roll control torque $u(t)$ is defined as $u/2 \triangleq u_1 = u_2$.

The transfer function from roll control torque to elastic deformations at various points on the flexible toroid can be obtained using the Laplace transform ($t \rightarrow s$) and the finite sine transform ($\psi \rightarrow n$) because of the periodicity in ψ. The finite sine transform is defined as

$$y(n,t) \triangleq \int_0^{\pi} y(\psi,t) \sin n\psi \, \mathrm{d}\psi \tag{4a}$$

$$y(\psi,t) \triangleq \frac{2}{\pi} \sum_{n=1}^{\infty} y(n,t) \sin n\psi \tag{4b}$$

The transformed equations of motion with the boundary conditions (3) incorporated can be written as

$$\left[n^4 + \frac{GJ}{EI} n^2 + \frac{\sigma R^4}{EI} s^2\right] \frac{1}{R} y(n,s) - \left(1 + \frac{GJ}{EI}\right) n^2 \theta(n,s)$$
$$= \frac{nR}{4EI} [(-1)^n - 1] \, u(s) \tag{5}$$

and

$$\left(1 + \frac{GJ}{EI}\right) n^2 \frac{1}{R} y(n,s) - \left(1 + \frac{GJ}{EI} n^2\right) \theta(n,s) = 0 \tag{6}$$

where the torsional inertia of the thin toroid is neglected (however, it can be easily included).

From Eqs. (5) and (6), we have

$$\frac{y(n,s)}{u(s)} = \frac{R^2}{EI} \frac{(n/2)[1 + (GJ/EI)n^2]}{\Delta(n,s)} \tag{7}$$

$$\frac{\theta(n,s)}{u(s)} = \frac{R}{EI} \frac{(n^3/2)[1 + (GJ/EI)]}{\Delta(n,s)} \tag{8}$$

where

$$\Delta(n,s) = \left(1 + \frac{GJ}{EI}\right)^2 n^4 - \left(n^4 + \frac{GJ}{EI} n^2 + \frac{\sigma R^4}{EI} s^2\right)\left(1 + \frac{GJ}{EI} n^2\right)$$

By the definition of the finite sine transform, we have

$$\frac{y(\psi,s)}{u(s)} = \frac{2R^2}{\pi EI} \sum_{n=1,3}^{\infty} \frac{(n/2)[1 + (GJ/EI)n^2]}{\Delta(n,s)} \sin n\psi \tag{9}$$

$$\frac{\theta(\psi,s)}{u(s)} = \frac{2R}{\pi EI} \sum_{n=1,3}^{\infty} \frac{(n^3/2)[1 + (GJ/EI)]}{\Delta(n,s)} \sin n\psi \tag{10}$$

The transfer function from roll control torque $u(s)$ to roll angle $\phi(s) \triangleq -y'(0,s)/R$ can be written as

$$\frac{\phi(s)}{u(s)} = \frac{1}{J} \sum_{n=1}^{\infty} \frac{n^2}{s^2 + \omega_n^2} \tag{11}$$

where $J \triangleq \sigma\pi R^3$ = roll moment of inertia of the rigid toroid.

The natural frequencies ω_n are defined as

$$\omega_n \triangleq \left[n^4 + \frac{GJ}{EI} n^2 - \frac{[1 + (GJ/EI)]^2 n^4}{1 + (GJ/EI)n^2}\right]^{1/2} \qquad n = 1,3,\ldots,\infty \tag{12}$$

where $\omega_1 = 0$ represents the rigid-body mode. It is seen that the residues, or modal gains, in Eq. (11) have the same sign. Thus, Eq. (11) will have alternating poles and zeros along the imaginary axis, which is a direct consequence of collocated actuator and sensor. For this case, attitude stabilization can be simply achieved using angle and rate feedback or attitude angle feedback with lead compensation.

For high-performance attitude controller design, the reduced-order transfer functions can be obtained from Eq. (11) by truncating the higher-frequency modes. The number of modes to be chosen depends on the control bandwidth and on the inherent damping of the structure.

Rigid Toroid with a Pretensioned Membrane

A simple model for roll attitude control of a rigid toroid with a pretensioned membrane is shown in Fig. 2. This is an approximate model for a space reflector with a reflector surface stretched across a rigid toroidal frame.

The equation of motion for a circular membrane in polar coordinates (r, ψ) is

$$z_{rr} + \frac{1}{r} z_r + \frac{1}{r^2} z_{\psi\psi} - \frac{1}{c^2} \ddot{z}(r,\psi,t) = 0 \tag{13}$$

where

$$(\ )_r \triangleq \partial(\ )/\partial r, \qquad (\ )_\psi \triangleq \partial(\ )/\partial\psi$$

and $z(r,\psi,t)$ is the transverse displacement of the uniform membrane, $c \triangleq (T_0/\rho)^{1/2}$, T_0 is the tension per unit length, and ρ is the mass per unit area of the membrane. The boundary conditions for pure roll motion are

$$z(r,0,t) = 0 \tag{14a}$$

$$z(r,\psi,t) = -(R \sin\psi) \, \phi(t) \tag{14b}$$

$$z_\psi(r,\pi/2,t) = 0 \tag{14c}$$

$$u(t) = J\ddot{\phi} - 4\int_0^{\pi/2} T_0 R \, [z_r(r,\psi,t) + \phi \sin\psi] R \sin\psi \, \mathrm{d}\psi \tag{14d}$$

where

$u(t)$ = roll control torque applied to the rigid toroid ($u/2 = u_1 = u_2$)
$\phi(t)$ = roll attitude angle of rigid toroid
J = $\sigma\pi R^3$ = moment of inertia of the rigid toroid

For a steady-state sinusoidal input, we have

$$u(t) = u(\omega)\sin\omega t \tag{15a}$$

$$\phi(t) = \phi(\omega)\sin\omega t \tag{15b}$$

From Eqs. (14a) and (14c), we have

$$z(r,\psi,t) = z(r,\omega)\sin\psi\sin\omega t \tag{16}$$

Substituting Eq. (16) into Eq. (13), we get

$$z_{rr} + \frac{1}{r}z_r + \left(\frac{\omega^2}{c^2} - \frac{1}{r^2}\right)z(r,\omega) = 0 \tag{17}$$

The solution of Eq. (17) is

$$z(r,\omega) = AJ_1(\omega r/c) + BY_1(\omega r/c) \tag{18}$$

where

$J_1(\omega r/c)$ = Bessel function of the first kind of order one
$Y_1(\omega r/c)$ = Bessel function of the second kind of order one

Since $z(0,\omega)$ is finite, we have $B = 0$. From Eq. (14a), we have

$$z(r,\omega) = -R\phi(\omega)\frac{J_1(\omega r/c)}{J_1(\omega R/c)} \tag{19}$$

Substituting Eq. (19) into Eq. (14d), we get the transfer function from roll control torque $u(s)$ to roll attitude angle $\phi(s)$,

$$\frac{\phi(s)}{u(s)} = \frac{J_1(\lambda)}{-\pi T_0 R^2[(K/4)\lambda^2 J_1(\lambda) + J_1(\lambda) - \lambda J_1'(\lambda)]} \tag{20}$$

where $s = j\omega = j(c/R)\lambda$ and

$$K \triangleq \frac{J}{(\pi/4)\rho R^4} = \frac{\text{moment of inertia of rigid toroid}}{\text{moment of inertia of rigid membrane}}$$

$$J_1'(\lambda) \triangleq \frac{\mathrm{d}J_1(\lambda)}{\mathrm{d}\lambda}$$

As $\lambda \to 0$, Eq. (20) becomes

$$\frac{\phi(s)}{u(s)} = \frac{1}{[J + (\pi/4)\rho R^4]s^2} \tag{21}$$

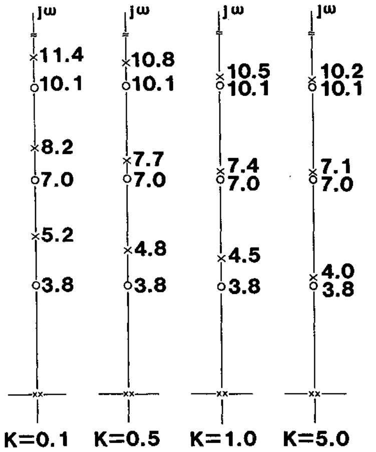

Fig. 3 Exact poles and zeros of collocated transfer function for rigid toroid with pretensioned membrane.

which is the transfer function of the rigid toroid with rigid membrane.

Pole-zero patterns for different values of K are shown in Fig. 3. As $\lambda \to \infty$, we have near pole-zero cancellations of all the vibration modes that correspond to the case of rigid toroid with negligible effects of membrane vibration.

The numerator of Eq. (20) is identical to the characteristic equation of circular membrane with fixed boundary and nodal line at the roll control axis. Thus, the zeros in Eq. (20) are independent of K.

Flexible Toroid with Pretensioned Membrane

In this section, we derive the equations of motion of the flexible toroid with a pretensioned membrane. The tension for out-of-plane buckling of the toroid and membrane will also be determined. Finally, the roll transfer function will be derived for the case of collocated actuator and sensor.

Equations of Motion

Consider a flexible toroid with a pretensioned membrane as shown in Fig. 4. The membrane equation is the same as Eq. (13)

$$z_{rr} + \frac{1}{r}z_r + \frac{1}{r^2}z_{\psi\psi} - \frac{1}{c^2}\ddot{z}(r,\psi,t) = 0$$

From the sketch of a toroid element of length $\mathrm{d}l$ shown in Fig. 4, we have

$$EI\left[\frac{\mathrm{d}\phi}{\mathrm{d}l} + \frac{\theta}{R}\right] = -Q \tag{22}$$

$$GJ\left[\frac{\mathrm{d}\theta}{\mathrm{d}l} - \frac{1}{R}\frac{\mathrm{d}y}{\mathrm{d}l}\right] = -T \tag{23}$$

where

y = transverse displacement of toroid
θ = torsional angle of toroid
ϕ = tangential bending slope of toroid
T = torsional torque
Q = bending torque
EI = bending stiffness
GJ = torsional stiffness

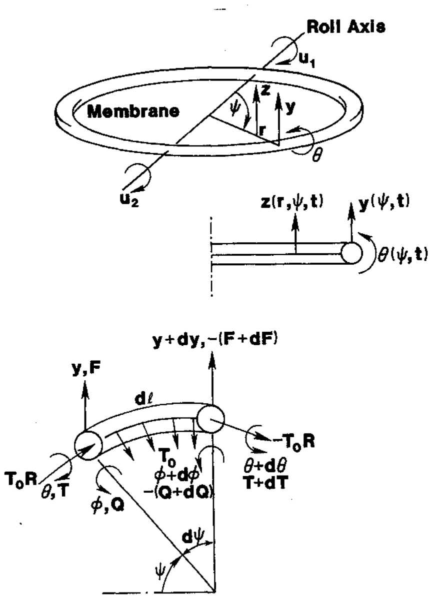

Fig. 4 Flexible toroid with pretensioned membrane.

By adding the transverse forces at the toroid element $\mathrm{d}l$, we get

$$\sigma\ddot{y} + T_0 z_r(r,\psi,t) = -\frac{\mathrm{d}F}{\mathrm{d}l} \tag{24}$$

where

σ = mass per unit length of toroid
T_0 = tension per unit length in membrane
F = transverse shear force

For the moments about the ϕ axis at the right end of the toroid element $\mathrm{d}l$, we have

$$\frac{\mathrm{d}Q}{\mathrm{d}l} + \frac{T}{R} - T_0 R\frac{\mathrm{d}y}{\mathrm{d}l} = -F \tag{25}$$

Similarly, for the moments about the θ axis at the right end, we have

$$J_0\ddot{\theta} = -\frac{\mathrm{d}T}{\mathrm{d}l} + \frac{Q}{R} - aT_0[\theta - z_r(R,\psi,t)] \tag{26}$$

where

J_0 = polar moment of inertia of toroid per unit length
$2a$ = thickness of toroid

Since $\phi = \mathrm{d}y/\mathrm{d}l$ and $\mathrm{d}l = R\mathrm{d}\psi$, combining the preceding equations gives

Toroid bending:

$$\frac{EI}{R^3}\left[\frac{1}{R}y'''' + \theta''\right] + \frac{GJ}{R^3}\left[\theta'' - \frac{1}{R}y''\right] + \frac{T_0}{R}y'' + \sigma\ddot{y} = -T_0\, z_r(R,\psi,t) \tag{27}$$

Toroid torsion:

$$\frac{GJ}{R^2}\left[\theta'' - \frac{1}{R}y''\right] - \frac{EI}{R^2}\left[\frac{1}{R}y'' + \theta\right] - aT_0\theta - J_0\ddot{\theta} = -aT_0\, z_r(R,\psi,t) \tag{28}$$

where $(\ )' \triangleq \partial(\ )/\partial\psi$. The boundary conditions for pure roll motion are

$$z(r,0,t) = z'(r,\pi/2,t) = 0$$
$$y(0,t) = y'(\pi/2,t) = 0$$
$$\theta(0,t) = \theta'(\pi/2,t) = 0$$
$$z(R,\psi,t) = y(\psi,t) - a\theta(\psi,t)$$
$$u(t) = \frac{4EI}{R}[y''(0,t)/R] \tag{29}$$

where $u(t)$ is the roll control torque defined as $u/2 = u_1 = u_2$. The equations of motion in dimensionless form can be written as

Toroid bending:

$$y'''' + \theta'' + k_1[\theta'' - y''] + k_2 y'' + \ddot{y} = -k_2 z_r(1,\psi,t) \tag{30}$$

Toroid torsion:

$$k_1[\theta'' - y''] - [y'' + \theta] - k_2k_5\theta - k_4\ddot{\theta} = -k_2k_5 z_r(1,\psi,t) \tag{31}$$

Membrane:

$$z_{rr} + \frac{1}{r}z_r + \frac{1}{r^2}z'' - \frac{k_3}{k_2}\ddot{z} = 0 \tag{32}$$

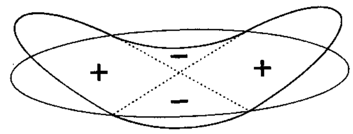

Fig. 5 Buckled mode shape of a thin flexible toroid with pretensioned membrane.

where

$$k_1 \triangleq \frac{GJ}{EI},\quad k_2 \triangleq \frac{T_0R^3}{EI},\quad k_3 \triangleq \frac{\rho R}{\sigma},\quad k_4 \triangleq \frac{J_0}{\sigma R^2},\quad k_5 \triangleq \frac{a}{R}$$

and (y,z,r) are in units of R, time in units of $[\sigma R^4/EI]^{1/2}$. The boundary conditions in dimensionless forms are

$$z(r,0,t) = z'(r,\pi/2,t) = 0$$
$$y(0,t) = y'(\pi/2,t) = 0$$
$$\theta(0,t) = \theta'(\pi/2,t) = 0$$
$$z(1,\psi,t) = y(\psi,t) - k_5\theta(\psi,t)$$
$$u(t) = 4y''(0,t) \tag{33}$$

where $u(t)$ is in units of EI/R. The equations of motion for a thin toroid ($k_4,k_5 \ll 1$) are

Toroid bending:

$$y'''' + (k_2 - k_1)y'' + (1 + k_1)\theta'' + \ddot{y} = -k_2 z_r(1,\psi,t) \tag{34}$$

Toroid torsion:

$$-(1 + k_1)y'' + k_1\theta'' - \theta = 0 \tag{35}$$

The membrane equation is the same as Eq. (32), and the boundary conditions are the same as Eq. (33) except that

$$z(1,\psi,t) = y(\psi,t) \tag{36}$$

Static Buckling Analysis

The critical tension (static buckling load) can be determined, assuming the buckled shape (see Fig. 5) as

$$y(\psi) = A\sin 2\psi$$
$$\theta(\psi) = B\sin 2\psi$$
$$z(r,\psi) = x(r)\sin 2\psi \tag{37}$$

Substituting Eq. (37) into the static membrane equation, we find that

$$z(r) = Ar^2 \tag{38}$$

which represents parabolic deflection of the membrane when the toroid buckles. Using Eqs. (37) and (38), and from the toroid bending and torsion equations of a thin toroid (static case), we can obtain the condition for the static out-of-plane buckling as

$$k_2 = 18k_1/(4k_1 + 1) \tag{39}$$

Since

$$k_1 = GJ/EI,\qquad k_2 = T_0R^3/EI$$

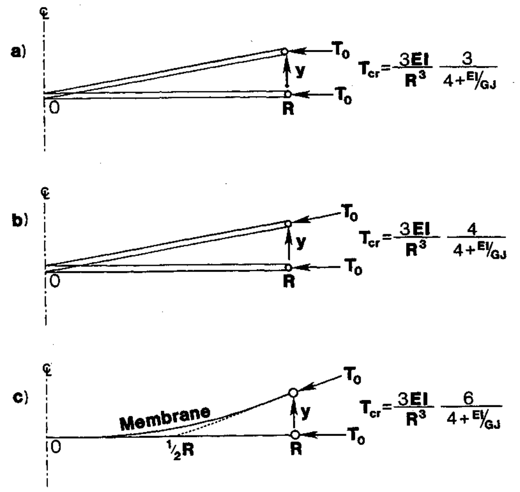

Fig. 6 Three different values of critical load for a thin flexible toroid.

the critical tension is expressed as

$$T_{cr} = \frac{3EI}{R^3} \frac{6}{4 + (EI/GJ)} \tag{40}$$

It is interesting to compare this result to other known values of critical loads for the different external loading conditions.

Case 1: Flexible ring (toroid) with externally applied constant uniform pressure (always in the same direction), as shown in Fig. 6a.

Case 2: Flexible ring with externally applied load (uniformly distributed) that always points to the center of the ring, as shown in Fig. 6b.

For case 1, Eqs. (34) and (35) can be modified to

$$y'''' + (k_2 - k_1)y'' + (1 + k_1)\theta'' = 0 \tag{41a}$$

$$-(1 + k_1)y'' + k_1\theta'' - \theta = 0 \tag{41b}$$

where there is no transverse component of external pressure in this case. By assuming the buckled shape, $y = A \sin 2\psi$ and $\theta = B \sin 2\psi$, we get the following critical value of external pressure:

$$T_{cr} = \frac{3EI}{R^3} \frac{3}{4 + (EI/GJ)} \tag{42}$$

which is exactly the same as the critical pressure obtained by Timoshenko in 1923.[10,11]

For case 2, Eqs. (34) and (35) can be modified to

$$y'''' + (k_2 - k_1)y'' + (1 + k_1)\theta'' = -k_2 y \tag{43a}$$

$$-(1 + k_1)y'' + k_1\theta'' - \theta = 0 \tag{43b}$$

Similarly, we get the critical load

$$T_{cr} = \frac{3EI}{R^3} \frac{4}{4 + (EI/GJ)} \tag{44}$$

which is identical to the critical load obtained by Hencky in 1921.[10,12]

Transfer Functions

The transfer functions from roll control torque to structural displacements at various points on the flexible thin toroid with a pretensioned membrane can be obtained by taking the Fourier transform ($t \to \omega$) and the finite sine transform ($\psi \to n$) of the coupled partial differential equations and the boundary conditions.

The transformed equation of motion of the membrane becomes

$$z_{rr} + \frac{1}{r} z_r + \left[-\frac{n^2}{r^2} + \frac{k_3}{k_2}\omega^2 \right] z(r,n,\omega) = 0 \tag{45}$$

Using the boundary conditions, the solution of Eq. (45) becomes

$$z(r,n,\omega) = \frac{y(n,\omega) J_n(\lambda r)}{J_n(\lambda)} \tag{46}$$

where $\lambda \triangleq \omega[k_3/k_2]^{1/2}$ and $J_n(\lambda)$ is the Bessel function of the first kind of order n. The toroid bending equation (34) after transformations becomes

$$\begin{aligned} &n^4 y(n,\omega) + n[(-1)^{n+1} y''(\pi,t) + y''(0,t)] \\ &\quad - n^3[(-1)^{n+1} y(\pi,t) + y(0,t)] + (k_2 - k_1)\{ -n^2 y(n,\omega) \\ &\quad + n[(-1)^{n+1} y(\pi,t) + y(0,t)]\} + (1 + k_1)\{ -n^2\theta(n,\omega) \\ &\quad + n[(-1)^{n+1}\theta(\pi,t) + \theta(0,t)]\} - \omega^2 y(n,\omega) \\ &\quad = -k_2 \frac{y(n,\omega)\lambda J_n'(\lambda)}{J_n(\lambda)} \end{aligned} \tag{47}$$

where

$$J_n'(\lambda) \triangleq \frac{dJ_n(\lambda)}{d\lambda}$$

Rearranging Eq. (47) and using $y''(0,\omega) = y''(\pi,\omega) = u(\omega)/4$, we get

$$\begin{aligned} &\left[n^4 + (k_1 - k_2)n^2 - \omega^2 + \frac{k_2 \lambda J_n'(\lambda)}{J_n(\lambda)} \right] y(n,\omega) \\ &\quad - (1 + k_1)n^2\theta(n,\omega) = \frac{n}{4}[(-1)^n - 1]u(\omega) \end{aligned} \tag{48}$$

Similarly, the toroid torsion equation (35) becomes

$$(1 + k_1)n^2 y(n,\omega) - (1 + k_1 n^2)\theta(n,\omega) = 0 \tag{49}$$

Combining Eqs. (48), (49), and (46), we get

$$\frac{y(n,\omega)}{u(\omega)} = \frac{(n/2)(1 + k_1 n^2) J_n(\lambda)}{\Delta(n,\omega)} \tag{50}$$

$$\frac{\theta(n,\omega)}{u(\omega)} = \frac{(n^3/2)(1 + k_1) J_n(\lambda)}{\Delta(n,\omega)} \tag{51}$$

$$\frac{z(r,n,\omega)}{u(\omega)} = \frac{(n/2)(1 + k_1 n^2) J_n(\lambda r)}{\Delta(n,\omega)} \tag{52}$$

where

$$\begin{aligned} \Delta(n,\omega) &\triangleq -[n^4 J_n(\lambda) + (k_1 - k_2)n^2 J_n(\lambda) - \omega^2 J_n(\lambda) \\ &\quad + k_2 \lambda J_n'(\lambda)](1 + k_1 n^2) + (1 + k_1)^2 n^4 J_n(\lambda) \end{aligned} \tag{53}$$

By using the definition of the finite sine transform given by Eqs. (4a) and (4b), we can write Eqs. (50–52) as

$$\frac{y(\psi,\omega)}{u(\omega)} = \frac{2}{\pi} \sum_{n=1,3}^{\infty} \frac{(n/2)(1 + k_1 n^2) J_n(\lambda)}{\Delta(n,\omega)} \sin n\psi \tag{54}$$

$$\frac{\theta(\psi,\omega)}{u(\omega)} = \frac{2}{\pi} \sum_{n=1,3}^{\infty} \frac{(n/2)(1 + k_1) J_n(\lambda)}{\Delta(n,\omega)} \sin n\psi \tag{55}$$

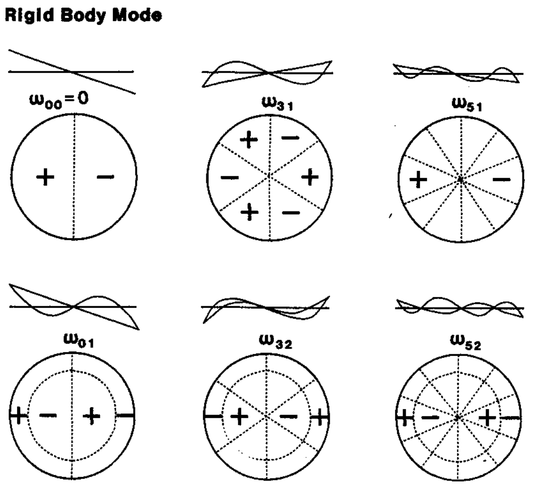

Fig. 7 Roll mode shapes of a flexible toroid with pretensioned membrane.

$$\frac{z(r,\psi,\omega)}{u(\omega)} = \frac{2}{\pi} \sum_{n=1,3}^{\infty} \frac{(n/2)(1+k_1 n^2) J_n(\lambda r)}{\Delta(n,\omega)} \sin n\psi \tag{56}$$

which are the transfer functions from the roll control torque to the displacements at various points on the toroid and membrane.

In particular, the transfer function from the roll control torque to the collocated roll angle sensor that measures the bending slope $\phi(\omega) \triangleq y'(0,\omega)$ can be written as

$$\frac{\phi(\omega)}{u(\omega)} = \frac{1}{\pi} \sum_{n=1,3}^{\infty} \frac{n^2(1+k_1 n^2) J_n(\lambda)}{\Delta(n,\omega)} \tag{57}$$

Using the Laplace transform variable s, the roll transfer function (57) can be rewritten as (in dimensional form)

$$\frac{\phi(s)}{u(s)} = \frac{1}{Js^2} + \sum_{n=1,3}^{\infty} a_n \prod_{m=1,2}^{\infty} \frac{1+(s/z_{nm})^2}{1+(s/\omega_{nm})^2} \tag{58}$$

where J = total moment of inertia of rigid toroid and rigid membrane, ω_{nm} = the mth root of the characteristic equation $\Delta(n,\omega)$ given by Eq. (53), z_{nm} = the mth root of $J_n(\lambda) = 0$. The a_n is defined as

$$a_1 = (1+k_1)/J \tag{59a}$$

$$a_n = \frac{n^2(1+k_1 n^2)}{J\{-[n^4 + (k_1 - k_2)n^2 + k_2](1+k_1 n^2) + (1+k_1)^2 n^4\}}$$

$$n = 3,5,\ldots,\infty \tag{59b}$$

The transfer function (57) or (58) is not as simple as Eq. (20) because of the combination of infinite product and infinite partial fraction expansions. This additional complexity is due to the coupling between two elastic systems, a flexible toroid and a membrane. Some of the roll mode shapes are shown in Fig. 7; there are an infinite number of mode shapes for each deflected shape of toroid (circular nodal lines).

Since Eq. (57) is the transfer function between collocated actuator and sensor, it has alternating poles and zeros along the imaginary axis. Unfortunately, we did not find the exact zeros, but Eq. (58) can be used to determine the reduced-order transfer function for the finite-dimensional controller design.

Conclusions

In this paper, we have attempted to develop generic models of a two-dimensional elastic system that are simple enough to treat analytically, yet complicated enough to demonstrate the dynamical characteristics of future space structures with a toroidal configuration. It was shown that the coupling between two elastic systems, a flexible toroid and a pretensioned membrane, results in a transfer function expressed as a combination of an infinite product expansion and an infinite partial fraction expansion. Such a representation clearly indicates the complicated dynamic behavior of the models studied in this paper.

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